Special Family of Ruled Surfaces in Euclidean 3-Space

Soukaina Ouarab, Amina Ouazzani Chahdi

Abstract— The purpose of the present paper is to construct a family of ruled surfaces with Frenet frame of an arbitray non-cylindrical and regular ruled surface in Euclidean 3-space. We study the most important characteristic properties of that special family of ruled surfaces such as the Guassian curvature, the mean curvature, the striction curve and give their associated characterizations. Moreover, we apply our study for special regular and non-cylindrical ruled surfaces whose striction curve is the unit circle, the general circular helix and the general non-circular helix, respectively.

Index Terms— Euclidean 3-space, Frenet frame, geodesic curvature, geodesic torsion, Gaussian curvature, mean curvature, normal curvature, ruled surface, striction curve,.

1 Introduction

In the classical differential geometry, the study of special surfaces and their properties is one of the most principal aims. The well-known types of special surfaces are helicoid, rotational, cyclic, canal, revolution, ruled surfaces...and are useful in many domains of applications, such as mathematical physics, moving geometry, kinematics and the domain of Computer Aided Geometric Design (CAGD).

Ruled surface [9] is one of the most special and fascinating surfaces which is defined by the moving of a straight line (ruling) around a curve (base curve). An important number of studies of ruled surfaces has been realized with special frames. Effectively, in [5], the authors have studied ruled surface with Frenet frame of a regular curve, they have investigated its properties and gave examples of ruled surfaces with Frenet frame of general and slant helices in euclidean 3-space. A few years ago, in [2,4], the authors have studied ruled surface with Darboux frame in euclidean 3-space, but in [6] it have been studied in Mikonwski 3-space. Recently, in [1] ruled surface and its properties have been studied with Alternative moving frame in euclidean 3-space where an important part has been investigated for examples of ruled surfaces with alternative moving frame of general and slant helices. Frenet frame of ruled surface [8], is one of the remarkable frames which has interested many researchers by the propreties that it is defined along the striction curve of a non-cylindrical ruled surface and it have been studied with different methods and with which many important new definitions and characterizations of ruled surfaces have been investigated such as slant ruled surface [3, 7].

In this work, we are interested in Frenet frame of ruled surface. Therefore, we construct and study special family of ruled surfaces with Frenet frame and which is defined precisely by a linear combination of Frenet frame vectors of an arbitrary non-cylindrical ruled surface in euclidean 3-space. Moreover, we present a part of applications of our study, in which we give examples with illustrations in the real Euclidean 3-space $\mathbb{R}^3$ of special ruled surfaces whose striction curve is the unit circle, the general circular helix and the general non-circular helix.

2 Preliminaries

Let be I and E^3 an open interval of $\mathbb{R}$ and an euclidean 3-space, respectively. A ruled surface ϕ is a special surface generated by a continuous moving of a line along a curve and has the parametrization

$$\phi(s,v)=c(s)+v\vec{q}(s),$$

where the curve $c=c(s)$ is called base curve of ruled surface and $q=q(s)$ is a unit direction vector of an oriented line in E^3 whose various positions are called rulings..
The unit normal vector denoted by $m=m(s,v)$ on ruled surface ϕ at a regular point $\phi(s,v)$ is given by

$$m(s,v)=(\phi_s\times\phi_v)/\|\phi_s\times\phi_v\|, \qquad (1)$$

Where $\phi_s=\partial\phi/\partial s$ and $\phi_v=\partial\phi/\partial v$.

The first I and the second II fundamental forms of ruled surface ϕ at a regular point $\phi(s,v)$ are defined respectively by

$$I(\phi_s ds+\phi_v dv)=Eds^2+2Fdsdv+Gdv^2,$$
$$II(\phi_s ds+\phi_v dv)=eds^2+2fdsdv+gdv^2,$$

Where $E=\|\phi_s\|^2, F=\langle\phi_s,\phi_v\rangle, G=\|\phi_v\|^2$,
and $e=\langle\phi_{ss},m\rangle, f=\langle\phi_{sv},m\rangle, g=\langle\phi_{vv},m\rangle=0.$

The Gaussian curvature K and the mean curvature H of ϕ at a regular point $\phi(s,v)$ are given respectively by:

$$K=-f^2/(EG-F^2),\ H=(Ge-2Ff)/2(EG-F^2).$$

Definition: A regular surface is developabele (resp. minimal) if its Gaussian curvature (resp. mean curvature) vanishes identically.

The normal curvature ρ_n the geodesic curvature ρ_g and the geodesic torsion θ_g of a regular curve on a regular surface are defined respectively by:

International Journal of Scientific & Engineering Research, Volume 10, Issue 5, May-2019
ISSN 2229-5518

$$\rho_n = \langle n, T' \rangle, \rho_g = \langle n \times T, T' \rangle, \theta_g = -\langle n \times T, n' \rangle,$$

where $\vec{n}$ and T are the unit normal of surface along the curve $c = c(s)$ and the unit tangent of the curve $c = c(s)$, respectively.

Definition: A regular curve on a regular surface is an asymptotic line (resp. geodesic curve, resp. principal line) if its normal curvature (resp. geodesic curvature, resp. geodesic torsion) vanishes identically.

Supposing that ϕ is a regular and non-cylindrcal ruled surface with $\|q\| = 1$. If v infinitely decreases, then along a ruling $s = s_0$, the unit normal m approaches a limiting direction. This direction is called the asymptotic normal (central tangent) direction denoted by $a(s_0)$ and from (1) defined by

$$a(s_0) = \lim_{v \to -\infty} m(s_0, v) = (q(s_0) \times q'(s_0)) / \|q'(s_0)\|.$$

The point at which the unit normal of ϕ is perpendicular to a is called the striction point (or central point) C and the set of striction points of all rulings is called striction curve of the ruled surface which we can denot by $\beta = \beta(s)$ and it is defined by

$$\beta(s) = c(s) - \left(\langle c'(s), q'(s) \rangle / \|q'(s)\|^2\right) q(s).$$

The vector h defined by $h = a \times q$ is called central normal vector. Then the orthonormal system $\{C; q, h, a\}$ is called Frenet frame of the ruled surface ϕ where C is the central point and q, h, a are unit vector of ruling, central normal and central tangent, respectively. The set of all bound vectors $q(s)$ at origin O constitutes a cone which is called directing cone of ruled surface ϕ. The end points of these unit vectors circumscribe a spherical curve k_1, called the spherical image of the ruled surface whose arc is denoted by s_1. The derivative formulae of Frenet frame of ruled surface ϕ with respect to the arc s are given by

$$q' = k_1 h, \quad h' = -k_1 q + k_2 a, \quad a' = -k_2 h,$$

Where $k_1 = ds_1 / ds$, $k_2 = ds_3 / ds$, and s_1, s_3 are the arcs of the spherical curves k_1, k_2 circumscribed by the bound vectors $q(s)$ and $a(s)$.

3 Family of Ruled Surfaces with Frenet of an Arbitrary Non-Cylindrical Ruled Surface in Euclidean 3-Space

Let be ϕ a non-cylindriclal and regular ruled surface of class $C^k (k \geq 3)$ in the euclidean 3-space E^3 defined by

$$\phi : (s, v) \in I \times R \mapsto \beta(s) + v q(s), \quad \|q(s)\| = 1, \qquad (2)$$

where $\beta = \beta(s)$ and $q = q(s)$ are a normal parmetrization of its striction curve and the directing vectors of its rulings, respectively.

Considering the family of ruled surfaces defined in E^3 by striction curve and Frenet frame vectors of ruled surface ϕ defined in (), as follows

$$\Psi(s, v) = \beta(s) + v(x_1 q(s) + x_2 h(s) + x_3 a(s)), \qquad (3)$$

Where x_1, x_2 and x_3 are three constants satisfying $x_1^2 + x_2^2 + x_3^2 = 1.$

$\beta = \beta(s)$ is the striction curve of ϕ, it verifies $< \beta', q' > = 0$, then $< \beta', h > = 0$, which means that β' belongs to the plan generated by both vectors q and a. Therefore, we can write

$$\beta' = \alpha_1 q + \alpha_2 a,$$

Where $\alpha_1 : s \in I \mapsto \alpha_1(s)$ and $\alpha_2 : s \in I \mapsto \alpha_2(s)$ are two real functions satisfying $\alpha_1^2 + \alpha_2^2 = 1.$

Differentiating (3) with respect to s and v, respectively, we get

$$\Psi_s = (\alpha_1 - v x_2 k_1) q + v(x_1 k_1 - x_3 k_2) h + (\alpha_2 + v x_2 k_2) a, \qquad (4)$$
$$\Psi_v = x_1 q + x_2 h + x_3 a, \qquad (5)$$

then, the unit normal denoted by $n = n(s, v)$ on the family of ruled surfaces defined in (3) at a regular point $\Psi(s, v)$ is given by

$$n = n(s, v) = (\Psi_s \times \Psi_v) / \|\Psi_s \times \Psi_v\| = X / \|X\|,$$

Where

$$X = (-x_2 \alpha_2 + vA) q + (x_1 \alpha_2 - x_3 \alpha_1 + vB) h + (x_2 \alpha_1 - vC) a,$$
$$\|X\| = \sqrt{(-x_2 \alpha_2 + vA)^2 + (x_1 \alpha_2 - x_3 \alpha_1 + vB)^2 + (x_2 \alpha_1 - vC)^2},$$

where $A = x_1 x_3 k_1 - (x_2^2 + x_3^2) k_2$, $B = x_2 (x_1 k_2 + x_3 k_1)$, $C = (x_1^2 + x_2^2) k_1 - x_1 x_3 k_2$.

From (4) and (5), the components E, F and G of the first fundamental form of Ψ at a regular point $\Psi(s, v)$ are given by:

$$E = 1 + 2 v x_2 (\alpha_2 k_2 - \alpha_1 k_1) + v^2 \left[x_2^2 (k_1^2 + k_2^2) + (x_1 k_1 - x_3 k_2)^2 \right]$$
$$F = x_1 \alpha_1 + x_3 \alpha_2, \quad G = 1.$$

Differentiating Ψ_s and Ψ_v in (4) and (5) with respect to s and v, we get

$$\Psi_{ss} = [\alpha_1' - v\{x_2 k_1' + k_1 (x_1 k_1 - x_3 k_2)\}] a$$
$$+ [k_1 \alpha_1 - k_2 \alpha_2 + v\{x_1 k_1' - x_3 k_2' - x_2 (k_1^2 + k_2^2)\}] h$$

- *Soukaina Ouarab is currently a PhD student in Mathematics (differential geometry) in Hassan II University of Casablanca, Morocco. E-mail: soukaina.ouarab.sma@gmail.com*
- *Amina Ouazzani Chahdi is currently a professor of higher education in Mathematics in Hassan II University of Casablanca, Morocco. E-mail: ouazzaniamina@hotmail.com*

International Journal of Scientific & Engineering Research, Volume 10, Issue 5, May-2019
ISSN 2229-5518

$$+\left[\alpha_2'+vx_2k_2'+vk_2\left(x_1k_1-x_3k_2\right)\right]a,$$

$$\Psi_{vs}=-x_2k_1q+\left(x_1k_1-x_3k_2\right)h+x_2k_2a,$$

Then, from these last and (1), the components e, f and g of the second fundamental form of Ψ at a regular point $\Psi(s,0)$ along its base curve $\beta=\beta(s)$ are given as follows

$$e(s,0)=\left[x_2\left(\alpha_1\alpha_2'-\alpha_2\alpha_1'\right)+\left(k_1\alpha_1-k_2\alpha_2\right)\left(x_1\alpha_2-x_3\alpha_1\right)\right]/R,$$

$$f(s,0)=\left[x_2^2\left(\alpha_2k_1+\alpha_1k_2\right)+\left(x_1k_1-x_3k_2\right)\left(x_1\alpha_2-x_3\alpha_1\right)\right]/R,$$

$$g(s,0)=0.$$

Where $R=\sqrt{x_2^2+\left(x_1\alpha_2-x_3\alpha_1\right)^2}$.

Consequently, from these last and the components of the first fundamental form, the Gaussian curvature K and the mean curvature H of the family of ruled surfaces defined in (3) at a regular point $\Psi(s,0)$ of the base curve $\beta=\beta(s)$ are investigated as follows:

$$K(s,0)=-\left[x_2^2\left(\alpha_2k_1+\alpha_1k_2\right)+\left(x_1k_1-x_3k_2\right)\left(x_1\alpha_2-x_3\alpha_1\right)\right]^2/R^4,$$

$$H(s,0)=\left[x_2^2\left(\alpha_1\alpha_2'-\alpha_2\alpha_1\right)+\left(k_1\alpha_2-k_2\alpha_2\right)\left(x_1\alpha_2-x_3\alpha_1\right)\right]/2R^3$$

$$-\left(x_1\alpha_1+x_3\alpha_2\right)\left[x_2^2\left(\alpha_2k_1+\alpha_1k_2\right)+\left(x_1k_1-x_3k_2\right)\left(x_1\alpha_2-x_3\alpha_1\right)\right]/R^3.$$

Corollary:
Ψ is developable along the base curve β if and only if the curvatures k_1 and k_2 satisfy the equation
$x_2^2\left(\alpha_2k_1+\alpha_1k_2\right)+\left(x_1k_1-x_3k_2\right)\left(x_1\alpha_2-x_3\alpha_1\right)=0.$

Corollary:
Ψ is minimal along the base curve β if and only if the curvatures k_1 and k_2 satisfy the equation

$$x_2\left(\alpha_1\alpha_2'-\alpha_2\alpha_1'\right)+\left(k_1\alpha_1-k_2\alpha_2\right)\left(x_1\alpha_2-x_3\alpha_1\right)$$

$$-2x_2^2\left(x_1\alpha_1+x_3\alpha_2\right)\left(\alpha_2k_1+\alpha_1k_2\right)$$

$$-2x_2^2\left(x_1\alpha_1+x_3\alpha_2\right)\left(x_1k_1-x_3k_2\right)\left(x_1\alpha_2-x_3\alpha_1\right)=0.$$

Another hand, denoting by n_0 and T the unit normal on Ψ at a regular curve point $\Psi(s,0)=\beta(s)$ and the unit tangent of $\beta=\beta(s)$ at the point $\beta(s)$.
We have

$$n_0=n(s,0)=\left[-x_2\alpha_2q+\left(x_1\alpha_2-x_3\alpha_1\right)h+x_2\alpha_2a\right]/R,$$

$$T=\alpha_1q+\alpha_2a.$$

From these last we get

$$n_0\times T=\alpha_2\left(x_1\alpha_2-x_3\alpha_1\right)q+x_2h-\alpha_1\left(x_1\alpha_2-x_3\alpha_2\right)a\Big/\sqrt{x_2^2+\left(x_1\alpha_2-x_3\alpha_2\right)^2}, \quad (6)$$

$$T'=\alpha_1'q+\left(\alpha_1k_1-\alpha_2k_2\right)h+\alpha_2'a, \quad (7)$$

$$n_0'=\left(Lq+Mh+Na\right)\Big/\left(x_2^2+\left(x_1\alpha_2-x_3\alpha_1\right)^2\right)^{3/2}, \quad (8)$$

$$L=-\left[x_2\alpha_2'+k_1\left(x_1\alpha_2-x_3\alpha_1\right)\right]\left(x_2^2+\left(x_1\alpha_2-x_3\alpha_1\right)^2\right)$$

$$+x_2\alpha_2\left(x_1\alpha_2'-x_3\alpha_1'\right)\left(x_1\alpha_2-x_3\alpha_1\right),$$

$$M=\left[-x_2\left(\alpha_2k_1+\alpha_1k_2\right)+x_1\alpha_2'-x_3\alpha_1'\right]\left(x_2^2+\left(x_1\alpha_2-x_3\alpha_1\right)^2\right)$$

$$-\left(x_1\alpha_2'-x_3\alpha_1'\right)\left(x_1\alpha_2-x_3\alpha_1\right)^2,$$

$$N=\left[k_2\left(x_1\alpha_2-x_3\alpha_1\right)+x_2\alpha_1'\right]\left(x_2^2+\left(x_1\alpha_2-x_3\alpha_1\right)^2\right)$$

$$-x_2\alpha_1\left(x_1\alpha_2'-x_3\alpha_1'\right)\left(x_1\alpha_2-x_3\alpha_1\right).$$

Consequently, from (6), (7) et (8), the normal curvature ρ_n, the geodesic curvature ρ_g and the geodesic torsion θ_g of $\beta=\beta(s)$ on ruled surfaces defined in (3) are investigated, respectively as follows:

$$\rho_n=\left[x_2\left(\alpha_1\alpha_2'-\alpha_2\alpha_1'\right)+\left(x_1\alpha_2-x_3\alpha_1\right)\left(\alpha_1k_1-\alpha_2k_2\right)\right]/R,$$

$$\rho_g=\left[x_1\alpha_1'+x_3\alpha_2+x_2\left(\alpha_1k_1-\alpha_2k_2\right)\right]/R,$$

$$\theta_g=-\left[\alpha_2\left(x_1\alpha_2-x_3\alpha_1\right)L+x_2M-\alpha_2\left(x_1\alpha_2-x_3\alpha_1\right)N\right]/R^4.$$

Where

Corollary:
$\beta=\beta(s)$ is an asymptotic line for Ψ if and only if
$x_2\left(\alpha_1\alpha_2'-\alpha_2\alpha_1'\right)+\left(x_1\alpha_2-x_3\alpha_1\right)\left(\alpha_1k_1-\alpha_2k_2\right)=0.$

Corollary:
$\beta=\beta(s)$ is a geodesic curve for Ψ if and only if
$\left(x_1\alpha_1+x_3\alpha_2\right)+x_2\left(\alpha_1k_1-\alpha_2k_2\right)=0.$

Corollary:
$\beta=\beta(s)$ is a principal line for Ψ if and only if
$\alpha_2\left(x_1\alpha_2-x_3\alpha_1\right)L+x_2M-\alpha_1\left(x_1\alpha_2-x_3\alpha_1\right)N=0.$

Another hand, when Ψ is non-cylindrical,i.e., $x_2^2\left(k_1^2+k_2^2\right)+\left(x_1k_1-x_3k_2\right)^2\neq0$, its striction curve denoted by $\mu=\mu(s)$ is obtained as follows:

$$\mu=\beta-\left[x_2\left(-\alpha_1k_1+\alpha_2k_2\right)/S\right]\left[x_1q+x_2h+x_3a\right],$$

Where $S=x_2^2\left(k_1^2+k_2^2\right)+\left(x_1k_1-x_3k_2\right)$.

Which implies the following corollaries:
Corollary:
β is the striction curve of Ψ if and only if
$-\alpha_1k_1+\alpha_2k_2=0.$

Corollary:
β is the striction curve of the family of ruled surfaces defined by $(s,v)\in I\times R\mapsto\beta(s)+v\left(x_1q+x_3a\right)$,
where $x_1^2+x_3^2=1$.

4 Applications

International Journal of Scientific & Engineering Research, Volume 10, Issue 5, May-2019
ISSN 2229-5518

In the following part, we presente study of some examples of the family of ruled surfaces which is defned in (3). We investigate its properties and give illustrations of some special ruled surfaces in the family studied using the MATLAB software. The .rst, the second and the third example are given with non-cylindircal and regular ruled surface whose striction curve is the unit circle, the circular helix and the general non-circular helix, respectively.

Example 1:
Considering the non-cylindrical and regular ruled surface ${}^1\phi=\left({}^1\phi_1,{}^1\phi_2,{}^1\phi_3\right)$ defined in $\mathbb{R}^3$ by the unit circle ${}^1\beta=(\cos s,\sin s,0)$ as its striction curve, as follows:

$$ {}^1\phi=\cos s+\left(v/\sqrt{2}\right)\sin s, $$

$$ {}^1\phi_2=\sin s-\left(v/\sqrt{2}\right)\cos s, $$

$$ {}^1\phi_3=v/\sqrt{2}. $$

Its Frenet frame $\left\{{}^1\beta(s);{}^1q(s),{}^1h(s),{}^1a(s)\right\}$ the associated curvatures ${}^1k_1$ and ${}^1k_2$ and the functions ${}^1\alpha_1$ and ${}^1\alpha_2$ are given, respectively by:

$$ q=\left(1/\sqrt{2}\right)(\sin s,-\cos s,1), $$

$$ h=(\cos s,\sin ss,0), $$

$$ a=\left(1/\sqrt{2}\right)(-\sin ss,\cos s,1), $$

$$ {}^1k_1={}^1k_2=1/\sqrt{2},{}^1\alpha_1=-1/\sqrt{2},{}^1\alpha_2=1/\sqrt{2}, $$

Hence, the special family studied ${}^1\Psi$ of ruled surfaces which is constructed with Frenet frame of ${}^1\phi$ and its characteristic properties are given respectively by

$$ {}^1\Psi=\left({}^1\Psi_1,{}^1\Psi_2,{}^1\Psi_3\right), \tag{9} $$

Where

$$ {}^1\Psi_1=(1+x_2v)\cos s, $$

$$ {}^1\Psi_2=\left[1+v(x_1-x_3)/\sqrt{2}\right]\sin s, $$

$$ {}^1\Psi_3=v(x_1+x_3)/\sqrt{2}. $$

$$ {}^1K(s,0)=-\left(x_1^2-x_3^2\right)^2/\left(2x_2^2+(x_1+x_3)^2\right)^2, $$

$$ {}^1H(s,0)=(x_1+x_3)\left((x_1-x_3)^2-1\right)/\left(2x_2^2+(x_1+x_3)^2\right)^{3/2}, $$

$$ {}^1\rho_n=-(x_1+x_3)/\sqrt{2x_2^2+(x_1+x_3)^2}, $$

$$ {}^1\rho_g=-\sqrt{2}x_2/\sqrt{2x_2^2+(x_1+x_3)^2}, $$

$$ \theta_g=0, $$

$$ {}^1\mu=\left({}^1\mu_1,{}^2\mu_1,{}^3\mu_1\right), $$

Where

$$ {}^1\mu_1=\cos s-2v\left[x_2(x_1-x_3)/\left(2x_2^2+(x_1-x_3)^2\right)\right]\left[(x_1-x_3)\sin s/\sqrt{2}+x_2\cos s\right], $$

$$ {}^1\mu_2=\sin s+2v\left[x_2(x_1-x_3)/\left(2x_2^2+(x_1-x_3)^2\right)\right]\left[(x_1-x_3)\cos s/\sqrt{2}+x_2\sin s\right], $$

$$ {}^1\mu_3=-\sqrt{2}vx_2(x_1+x_3)/\left(2x_2^2+(x_1-x_3)^2\right). $$

In the following, some illustrations of some special ruled surfaces which are defined in the family (9)

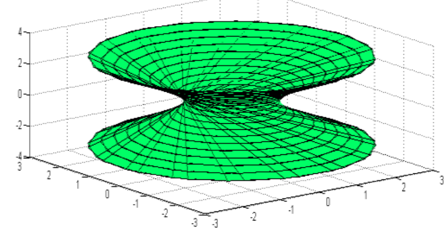

Fig. 1. Ruled surface ${}^1\Psi$ with $x_1=1,x_2=x_3=0$.

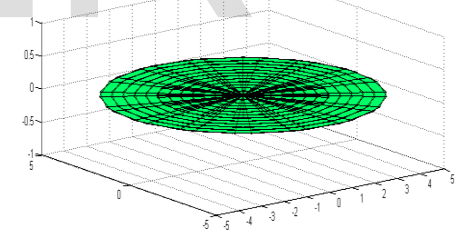

Fig. 2. Ruled surface ${}^2\Psi$ with $x_2=1,x_1=x_3=0$.

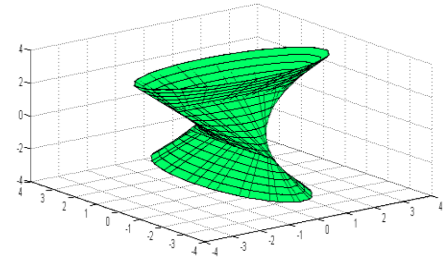

Fig. 3. Ruled surface ${}^1\Psi$ with $x_1=x_2=x_3=1/\sqrt{3}.$

International Journal of Scientific & Engineering Research, Volume 10, Issue 5, May-2019
ISSN 2229-5518

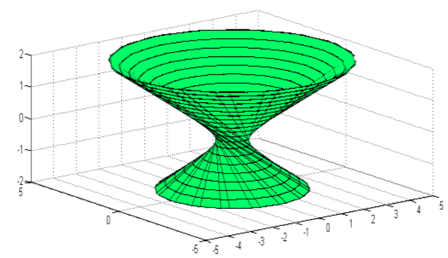

Fig. 4. Ruled surface ${}^1\Psi$ with $x_1 = x_2 = 1/\sqrt{2}, x_3 = 0$.

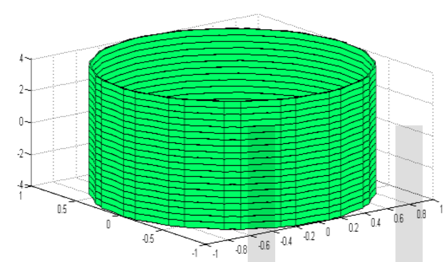

Fig. 5. Ruled surface ${}^1\Psi$ with $x_1 = x_3 = 1/\sqrt{2}, x_2 = 0.$

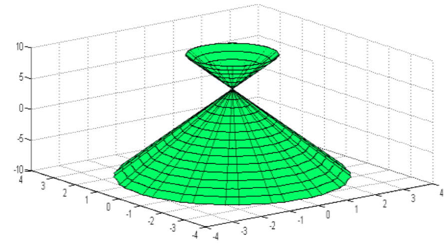

Fig. 6. Ruled surface ${}^1\Psi$ with $x_1 = x_3 = 1/\sqrt{3}, x_2 = -1/\sqrt{3}$.

These above figures are ploted with $(s,v) \in]-2\Pi, 2\Pi[\times]-4,4[$.

Example 2:
Considering the non-cylindrical and regular ruled surface ${}^2\phi = \left({}^2\phi_1, {}^2\phi_2, {}^2\phi_3\right)$ in ℝ³ defined with the general helix

$$ {}^2\beta(s) = (1/2)\left(\sin\left(\sqrt{2}s\right), -\cos\left(\sqrt{2}s\right), \sqrt{2}s\right), $$

as its striction curve as follows:

$$ {}^2\phi_1 = (1/2)\sin\left(\sqrt{2}s\right) - v\cos\left(\sqrt{2}s\right)/\sqrt{2}, $$
$$ {}^2\phi_2 = -(1/2)\cos\left(\sqrt{2}s\right) - v\sin\left(\sqrt{2}s\right)/\sqrt{2}, $$
$$ {}^2\phi_3 = (s+v)/\sqrt{2}. $$

Its Frenet frame $\left\{{}^2\beta(s); {}^2q(s), {}^2h(s), {}^2a(s)\right\}$, the associated curvatures ${}^2k_1$ and ${}^2k_2$ and the functions ${}^2\alpha_1$ and ${}^2\alpha_2$ are given, respectively by

$$ {}^2q = 1/\sqrt{2}\,(-\cos s, -\sin s, 1), $$
$$ {}^2h = \left(\sin\left(\sqrt{2}s\right), -\cos\left(\sqrt{2}s\right), 0\right), $$
$$ {}^2a = 1/\sqrt{2}\left(\cos\left(\sqrt{2}s\right), \sin\left(\sqrt{2}s\right), 1\right), $$
$$ {}^2k_1 = {}^2k_2 = 1, \ {}^2\alpha_1 = 0, \ {}^2\alpha_2 = 1. $$

Hence, the studied family of ruled surfaces, which is constructed with Frenet frame of ${}^2\phi$ and its characteristic properties are investigated, respectively by

$$ {}^2\Psi = \left({}^2\Psi_1, {}^2\Psi_2, {}^2\Psi_3\right), \tag{10} $$

Where

$$ {}^2\Psi_1 = (1/2)\sin\left(\sqrt{2}s\right) + v\left[(-x_1 + x_3)\cos\left(\sqrt{2}s\right)/\sqrt{2} + x_2\sin\left(\sqrt{2}s\right)\right], $$
$$ {}^2\Psi_2 = -(1/2)\cos\left(\sqrt{2}s\right) + v\left[(-x_1 + x_3)\sin\left(\sqrt{2}s\right)/\sqrt{2} - x_2\cos\left(\sqrt{2}s\right)\right], $$
$$ {}^2\Psi_3 = \left(s/\sqrt{2}\right) + v(x_1 + x_3)/\sqrt{2}. $$
$$ {}^2K(s,0) = -\left[\left(x_2^2 + x_1^2 - x_3x_1\right)/\left(x_2^2 + x_1^2\right)\right]^2, $$
$$ {}^2H(s,0) = -\left[x_1 + 2x_3\left(x_2^2 + x_1^2 - x_3x_1\right)\right]/\left[2\left(x_2^2 + x_1^2\right)^{3/2}\right], $$
$$ {}^2\rho = -x_1/\sqrt{x_2^2 + x_1^2}, $$
$$ \rho_g = -x_2/\sqrt{x_2^2 + x_1^2}, $$
$$ {}^3\theta = 1, $$
$$ {}^2\mu = \left({}^2\mu_1, {}^2\mu_2, {}^2\mu_3\right), $$

Where

$$ {}^2\mu_1 = {}^2\beta_1 - x_2\left[(-x_1 + x_3)\cos\left(\sqrt{2}s\right)/\sqrt{2} + x_2\sin\left(\sqrt{2}s\right)\right]/\left[2x_2^2 + (x_1 - x_3)^2\right], $$
$$ {}^2\mu_2 = {}^2\beta_2 - x_2\left[(-x_1 + x_3)\sin\left(\sqrt{2}s\right)/\sqrt{2} - x_2\cos\left(\sqrt{2}s\right)\right]/\left[2x_2^2 + (x_1 - x_3)^2\right], $$
$$ {}^2\mu_3 = {}^2\beta_3 - x_2(x_1 + x_3)/\left[2\sqrt{2}x_2^2 + \sqrt{2}(x_1 - x_3)^2\right]. $$

In the following, some illustrations of some special ruled surfaces which are defined in the family (10)

International Journal of Scientific & Engineering Research, Volume 10, Issue 5, May-2019
ISSN 2229-5518

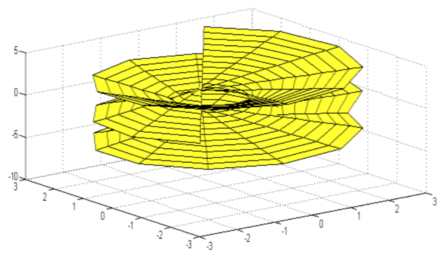

Fig. 7. Ruled surface ${}^{2}\Psi$ with $x_1 = 1, x_2 = x_3 = 0$.

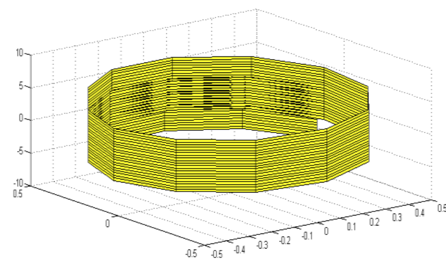

Fig. 10. Ruled surface ${}^{2}\Psi$ with $x_1 = x_3 = 1/\sqrt{2}, x_2 = 0$.

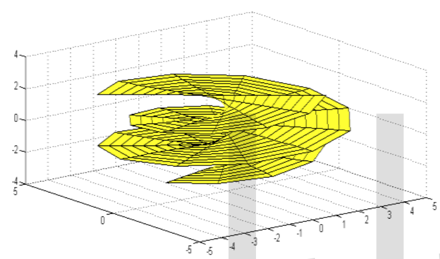

Fig. 8. Ruled surface ${}^{2}\Psi$ with $x_2 = 1, x_1 = x_3 = 0$.

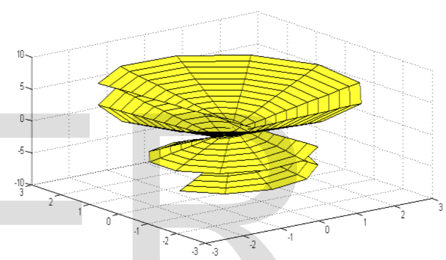

Fig. 11. Ruled surface ${}^{2}\Psi$ with $x_1 = x_2 = x_3 = 1/\sqrt{3}$.

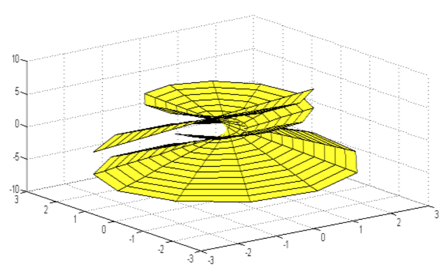

Fig. 9. Ruled surface ${}^{2}\Psi$ with $x_1 = x_3 = 1/\sqrt{3}, x_2 = -1/\sqrt{3},$.

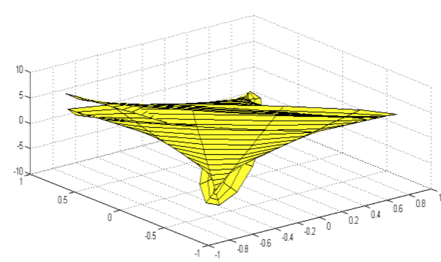

Fig. 12. Ruled surface ${}^{2}\Psi$ with $x_1 = 1/\sqrt{3}, x_2 = 0, x_3 = \sqrt{2/3}$.

These above figures are ploted with $(s,v) \in]-\Pi,\Pi[\times]-4,4[$, and the following values of the constants

International Journal of Scientific & Engineering Research, Volume 10, Issue 5, May-2019
ISSN 2229-5518

Example 3:

Considering the non-cylindrical and regular ruled surface ${}^3\phi = \left({}^3\phi_1, {}^3\phi_2, {}^3\phi_3\right)$ defined with the non-circular general helix ${}^3\beta = \left({}^3\beta_1, {}^3\beta_2, {}^3\beta_3\right)$ as its base and striction curve as follows:

$$ {}^3\phi_1 = 2s[\cos(2\ln s)/2 + \sin(2\ln s)]/5\sqrt{5} - 2v\cos(2\ln s)/\sqrt{5}, $$

$$ {}^3\phi_2 = 2s[\sin(2\ln s)/2 - \cos(2\ln s)]/5\sqrt{5} - 2v\sin(2\ln s)/\sqrt{5}, $$

$$ {}^3\phi_3 = 2s/\sqrt{5} + v/\sqrt{5}. $$

Its Frenet frame $\{{}^3\beta(s); {}^3q(s), {}^3h(s), {}^3a(s)\}$, the associated curvatures ${}^3k_1$ and ${}^3k_2$ and the functions ${}^3\alpha_1$ and ${}^3\alpha_2$ are given respectively by

$$ {}^3q = (1/\sqrt{5})(-2\cos(2\ln s) - 2\sin(2\ln s), 1), $$

$$ {}^3h = (\sin(2\ln s), -\cos(2\ln s), 0), $$

$$ {}^3a = (1/\sqrt{5})(\cos(2\ln s), \sin(2\ln s), 2), $$

$$ {}^3k_1 = 4/(\sqrt{5}s), \; {}^3k_2 = 2/(\sqrt{5}s), \; {}^3\alpha_1 = 0, \; {}^3\alpha_2 = 1. $$

Hence, the studied family of ruled surfaces which is constructed with Frenet frame of ${}^3\phi$ and its characteristic properties are investigated, respectively by

$$ {}^3\Psi = \left({}^3\Psi_1, {}^3\Psi_2, {}^3\Psi_3\right), \qquad (11) $$

Where

$$ {}^3\Psi_1 = (2s/5\sqrt{5})[\cos(2\ln s)/2 + \sin(2\ln s)] + v[(x_3 - 2x_1)\cos(2\ln s)/\sqrt{5} + x_2\sin(2\ln s)], $$

$$ {}^3\Psi_2 = (2s/5\sqrt{5})[\sin(2\ln s)/2 - \cos(2\ln s)] + v[(x_3 - 2x_1)\sin(2\ln s)/\sqrt{5} - x_2\cos(2\ln s)], $$

$$ {}^3\Psi_3 = 2s/\sqrt{5} + v(x_1 + 2x_3)/\sqrt{5}. $$

$$ {}^3K(s,0) = -(4/5s^2)\left[2x_2^2 + x_1(2x_1 - x_3)\right]^2 / \left[x_2^2 + x_1^2\right]^2, $$

$$ {}^3H(s,0) = -x_1 + \left[2x_3\left(2x_2^2 + x_1(2x_1 - x_3)\right)\right] / \left[\sqrt{5}s\left(x_2^2 + x_1^2\right)^{3/2}\right], $$

$$ {}^3\rho_n = -2x_1/\sqrt{5}\left[s\left(x_2^2 + x_1^2\right)\right], $$

$$ {}^3\rho_g = -2x_2/\sqrt{5}\left[s\left(x_2^2 + x_1^2\right)\right], $$

$$ {}^3\theta_g = 4/(\sqrt{5}s), $$

$$ {}^3\mu = \left({}^3\mu_1, {}^3\mu_2, {}^3\mu_3\right), $$

Where

$$ {}^3\mu_1 = {}^3\beta_1 - \left[x_2\sqrt{5}s/\left(10x_2^2 + 2(2x_1 - x_3)^2\right)\right]\left[(x_3 - 2x_1)\cos(2\ln s)/\sqrt{5} + x_2\sin(2\ln s)\right], $$

$$ {}^3\mu_2 = {}^3\beta_2 - \left[x_2\sqrt{5}s/\left(10x_2^2 + 2(2x_1 - x_3)^2\right)\right]\left[(x_3 - 2x_1)\sin(2\ln s)/\sqrt{5} - x_2\cos(2\ln s)\right], $$

$$ {}^3\mu_3 = {}^3\beta_3 - \left[x_2 s/\left(10x_2^2 + 2(2x_1 - x_3)^2\right)\right](x_1 + 2x_3), $$

In the following, some illustrations of some special ruled surfaces which are defined in the family (11)

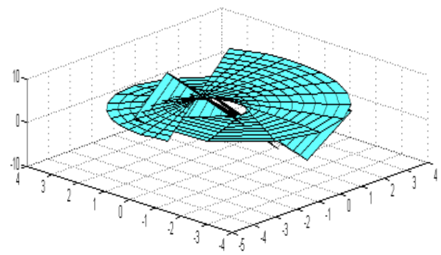

Fig. 13. Ruled surface ${}^3\Psi$ with $x_1 = 1, x_2 = x_3 = 0$.

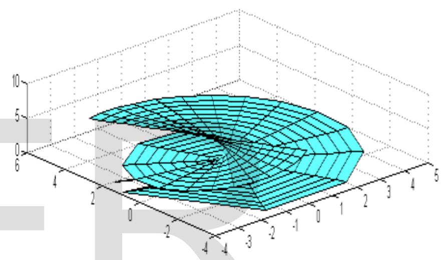

Fig. 14. Ruled surface ${}^3\Psi$ with $x_2 = 1, x_1 = x_3 = 0$.

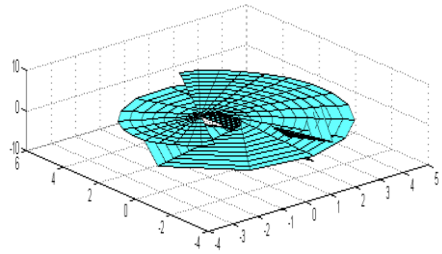

Fig. 15. Ruled surface ${}^3\Psi$ with $x_1 = 1/\sqrt{3}, x_2 = 0, x_3 = \sqrt{2/3}$.

International Journal of Scientific & Engineering Research, Volume 10, Issue 5, May-2019
ISSN 2229-5518

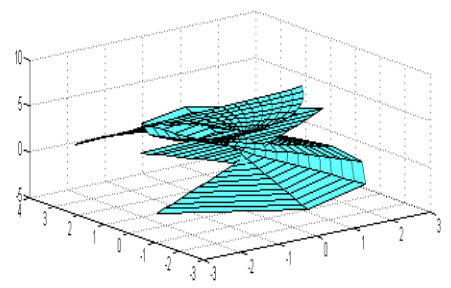

Fig. 16. Ruled surface ${}^{3}\Psi$ with $x_1 = x_3 = 1/\sqrt{3}, x_2 = -1/\sqrt{3}$.

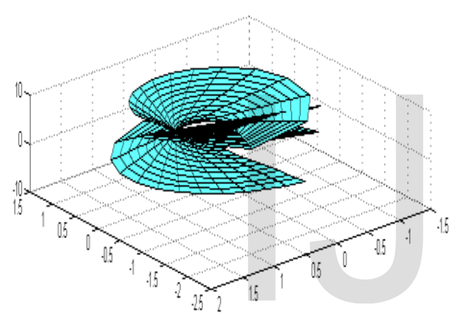

Fig. 17. Ruled surface ${}^{3}\Psi$ with $x_1 = x_3 = 1/\sqrt{2}, x_2 = 0$.

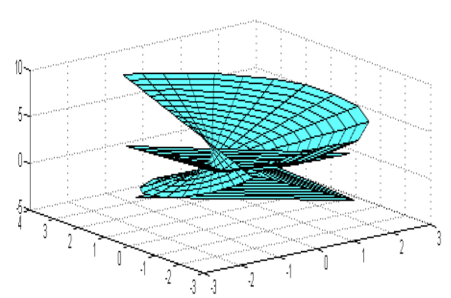

Fig. 18. Ruled surface ${}^{2}\Psi$ with $x_1 = x_2 = x_3 = 1/\sqrt{3}$.

These above figures are ploted with $(s,v) \in \left]\Pi/10^3, 2\Pi\right[\times \left]-4,4\right[$.

References

[1] S. Ouarab, A. Ouazzani, M. Izid, "*Ruled Surfaces with Alternative moving frame in Euclidean 3-space*", International J. of Math. Sci. & Engg. Appls. (IJMSEA), 43-58, Decem, 2018.

[2] Y. Unluturk, M. Cimdiker, C. Ekici, "*Characteristic properties of the parallel ruled surfaces with Darboux frame in Euclidean 3- space*", CMMA 1, No. 1 26-43, 2016.

[3] M. Onder , O. Kaya, "*Darboux Slant Ruled Surfaces*", Azerbaijan Journal of Mathematics, 2015

[4] G. Y. Sentürk, S. Yüce, "*Characteristic properties of the ruled surface with Darboux frame in* E^3 ", Kuwait J. Sci. 42 (2) 14-33, 2015.

[5] H. T. Ali, H. S. Abdel Aziz, A. H. *"Sorour, Ruled surfaces generated by some special curves in Euclidean 3-Space",* J. Egyptian. Math. Soc 21, 285-294, 2013

[6] S. Kiziltug, A. çakmak, "*Developable Ruled Surfaces with Darboux Frame in Minkowski 3-space* E^3 ", Life Science Journal, 10 (4), 2013

[7] M. Onder, "*Slant Ruled Surfaces in Euclidean 3-space*", arXiv:1311.0627v1 [math.DG], 2013.

[8] A. Karger, J. Novak, "*Sp ace Kinematics and Lie Groups*", STNL Publishers of Technical Lit, Czechoslovakia, 1978.

[9] P. do-carmo, "*Differential Geometry of curves and surfaces*", IMPA, 511, 1976.